

Hyperbolic sine and cosine

The hyperbolic sine is defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad (1)$$

where $x \in \mathbb{R}$ is a *real* variable.

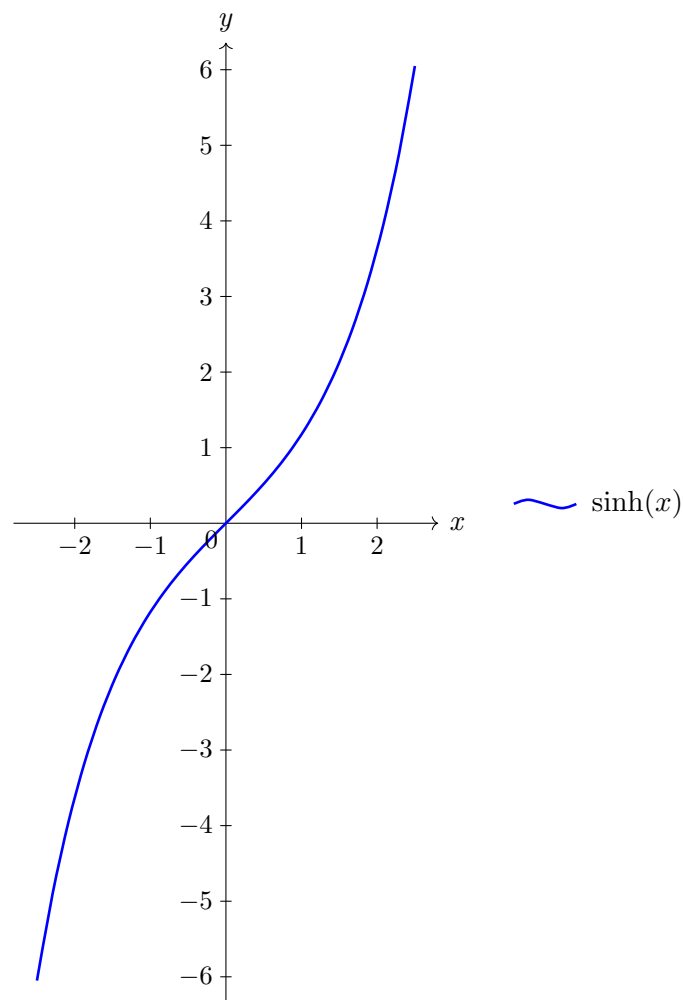


Figure 1: Hyperbolic sine as $x \in [-2.5, 2.5]$.

The hyperbolic cosine is defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad (2)$$

$x \in \mathbb{R}$.

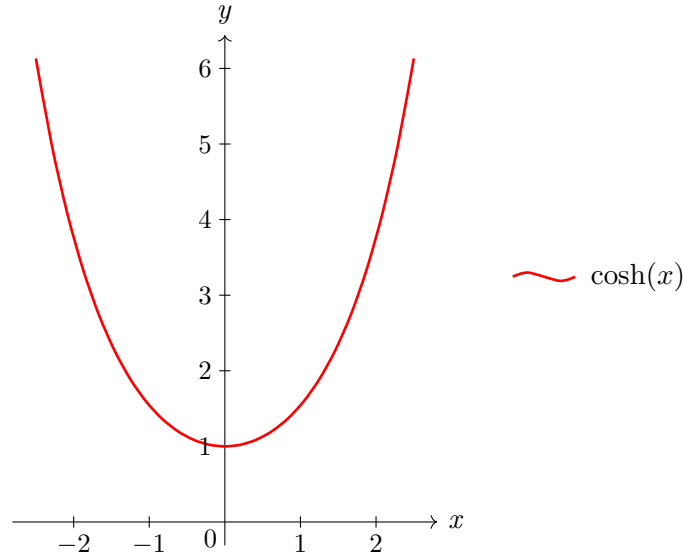


Figure 2: Hyperbolic cosine as $x \in [-2.5, 2.5]$.

The quantities they represent are the hyperbolic correspondants of the trigonometric quantities $\sin(x)$ and $\cos(x)$. There are some similarities and some differences between these functions.

As $\sin(x)$, $\sinh(x)$ is an odd function. As $\cos(x)$, $\cosh(x)$ is an even function.

The mutual behaviour of $\sin(x)$ and $\cos(x)$ remains unchanged for every real value of x (it is always $\sin(x + \pi/2) = \cos(x)$), but this is not the case for the hyperbolic functions.

$\sinh(x)$ and $\cosh(x)$ have different paths as $x \rightarrow 0$. Moreover,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sinh(x) &= \lim_{x \rightarrow +\infty} \cosh(x) = \frac{e^x}{2} \\ \lim_{x \rightarrow -\infty} \sinh(x) &= - \lim_{x \rightarrow +\infty} \cosh(x) = -\frac{e^{-x}}{2} \end{aligned} \quad (3)$$

They tend to coincide¹ for $x \rightarrow +\infty$; they tend instead to be symmetrical with respect to the x -axis as $x \rightarrow -\infty$, so that their *absolute values* still coincide², because the e^x terms tend to disappear.

Note that in both the limits the exponents of e are positive: in the first case, x is positive; in the second case, $-x$ is the result of a negative quantity x preceded by a negative sign.

	$\sinh(x)$	$\cosh(x)$
$\lim_{x \rightarrow +\infty}$	$\frac{e^x}{2}$	$\frac{e^x}{2}$
$\lim_{x \rightarrow -\infty}$	$-\frac{e^{-x}}{2}$	$\frac{e^{-x}}{2}$

Table 1: Asymptotic behaviours comparison.

The trigonometric functions are always bounded: $-1 \leq \sin(x) \leq 1$, $-1 \leq \cos(x) \leq 1$, $\forall x \in \mathbb{R}$. Their image is always the $[-1, 1]$ subset of \mathbb{R} .

The image of $\sinh(x)$ is instead the whole \mathbb{R} , and the image of $\cosh(x)$ is the subset $[1, +\infty)$ of \mathbb{R} . The hyperbolic cosine can never be smaller than 1.

Complex sine and cosine

When considering a *complex* variable $z \in \mathbb{C}$, the trigonometric sine is defined as:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad (4)$$

while the trigonometric cosine is defined as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad (5)$$

There is a strong similarity between (4) and (1), as well as between (2) and (5).

¹The terms $\pm e^{-x}$ tend to disappear and the two hyperbolic functions are overlapped in the limit case.

²The path of $\cosh(x)$ in fact overlaps $|\sinh(x)|$.



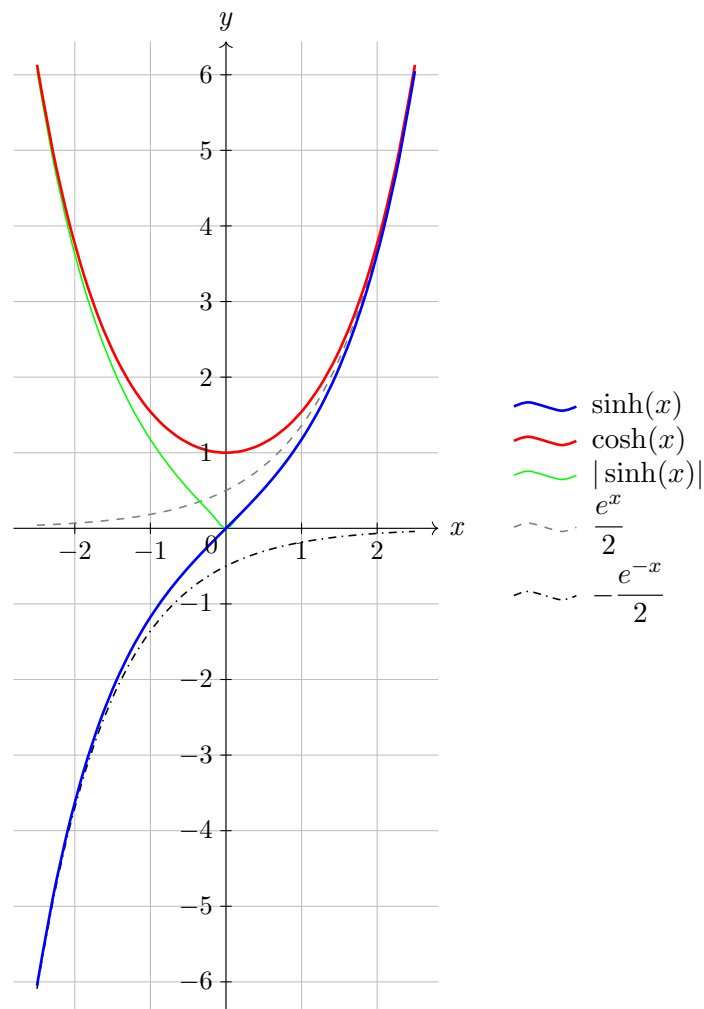


Figure 3: Graphical comparison of the asymptotic behaviours of $\sinh(x)$ and $\cosh(x)$, according to the limits and the Table 1.

In particular, if z is pure imaginary, so that $z = ix, x \in \mathbb{R}$, the obtained values are:

$$\begin{aligned}\sin(z) &= \sin(ix) = \frac{e^{i \cdot (ix)} - e^{-i \cdot (ix)}}{2i} = \frac{e^{-x} - e^x}{2i} = -\frac{\sinh(x)}{i} = i \sinh(x) \\ \cos(z) &= \cos(ix) = \frac{e^{i \cdot (ix)} + e^{-i \cdot (ix)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh(x)\end{aligned}\tag{6}$$

The trigonometric sine of a *pure imaginary* quantity $z = ix$ is i times the hyperbolic sine of x : it is a pure imaginary quantity itself, being $\sinh(x) \in \mathbb{R}$.

The trigonometric cosine of a *pure imaginary* quantity $z = ix$ is the hyperbolic cosine of x . Note that $\cosh(x) \in \mathbb{R}$: so, given a pure imaginary angle ix , its cosine is a real quantity.

Real values greater than unity

While $\sinh(x)$ may assume any real value, included values such that $|\sinh(x)| \geq 1$, it is always $\cosh(x) \geq 1$. So, not only the $\cos(z)$ may exceed 1 when $z = ix$ is pure imaginary, but it can never be less than 1.

To obtain a $\sin(z) \geq 1$ as well, an appropriate complex³ z can be chosen such that $\sin(z)$ equals the cosine of a pure imaginary quantity, which is – so far – the only suitable way to obtain a real quantity not smaller than 1.

Let S be the desired real quantity to be obtained from the sine. $S \geq 1$. This can be accomplished in two ways.

³The quantity z can not be real, neither pure imaginary, because it would respectively lead to $-1 \leq \sin(z) \leq 1$ and to (6).



Method 1

Choose

$$z = i \log \left[-i \left(S + \sqrt{S^2 - 1} \right) \right] \quad (7)$$

where $\log[\dots]$ is the natural logarithm. This value for z can be obtained by a simple inspection of the definition (4) of sine, such that its expression will finally be equal to S only.

To prove (7):

$$\begin{aligned} \sin(z) &= \sin \left\{ i \log \left[-i \left(S + \sqrt{S^2 - 1} \right) \right] \right\} = \\ &= \frac{e^{-\log \left[-i \left(S + \sqrt{S^2 - 1} \right) \right]} - e^{\log \left[-i \left(S + \sqrt{S^2 - 1} \right) \right]}}{2i} = \\ &= \frac{1}{-i \left(S + \sqrt{S^2 - 1} \right) + i \left(S + \sqrt{S^2 - 1} \right)} = \quad (8) \\ &= \frac{1 + S^2 + 2S\sqrt{S^2 - 1} + S^2 - 1}{-i \left(S + \sqrt{S^2 - 1} \right)} = \\ &= \frac{2S \left(S + \sqrt{S^2 - 1} \right)}{-i \left(S + \sqrt{S^2 - 1} \right)} \cdot \frac{1}{2i} = S \end{aligned}$$

An alternative proof can be made after some preliminary observations. It is worth noting in fact that, remembering the definition of the logarithm of a pure imaginary number:

$$i = e^{i\frac{\pi}{2}} \tag{9}$$

$$\log i = i\frac{\pi}{2}$$

and also the fact that $-i = 1/i$ and the properties of logarithms,

$$\begin{aligned} \log [-i (S + \sqrt{S^2 - 1})] &= \log \left[\frac{(S + \sqrt{S^2 - 1})}{i} \right] = \\ &= \log (S + \sqrt{S^2 - 1}) - \log i = \\ &= \log (S + \sqrt{S^2 - 1}) - i\frac{\pi}{2} \end{aligned} \tag{10}$$

the complex angle z in (7) can be rewritten as

$$\begin{aligned} z &= i \log [-i (S + \sqrt{S^2 - 1})] = \\ &= i \cdot \left[\log (S + \sqrt{S^2 - 1}) - i\frac{\pi}{2} \right] = \\ &= \frac{\pi}{2} + i \log (S + \sqrt{S^2 - 1}) \end{aligned} \tag{11}$$

It has been stated that $\sin(z)$ in (8) can provide a real value $S \geq 1$ only if z is such that the sine can be rewritten as a cosine of a pure imaginary quantity. This *is* the case: in fact,

$$\sin \left(\frac{\pi}{2} + \alpha \right) = \cos(\alpha) \tag{12}$$

for any $\alpha \in \mathbb{C}$ and, in this case, in particular for $\alpha = ix$. Comparing (12) with (11):

$$\sin \left\{ \frac{\pi}{2} + i \log [S + \sqrt{S^2 - 1}] \right\} = \cos \left[i \log (S + \sqrt{S^2 - 1}) \right] \tag{13}$$



Being $S \geq 1$, the square root is a real quantity, as well as the argument of the logarithm, and (therefore) its value. So, from (6),

$$\begin{aligned} \cos [i \log (S + \sqrt{S^2 - 1})] &= \cos(ix) = \\ &= \cosh(x) = \cosh [\log (S + \sqrt{S^2 - 1})] \end{aligned} \tag{14}$$

This seems not to immediately lead to the desired result S . However, remembering the definition of hyperbolic inverse cosine,

$$\operatorname{arccosh}(y) = \log (y + \sqrt{y^2 - 1}) \tag{15}$$

expression (14) is

$$\begin{aligned} \cosh [\log (S + \sqrt{S^2 - 1})] &= \\ &= \cosh [\operatorname{arccosh}(S)] = S \end{aligned} \tag{16}$$

which is the same value already obtained in (8).

Summarizing, from equations (13)-(16):

$$\begin{aligned} \sin \left\{ \frac{\pi}{2} + i \log [S + \sqrt{S^2 - 1}] \right\} &= \\ &= \cos [i \log (S + \sqrt{S^2 - 1})] = \\ &= \cosh [\log (S + \sqrt{S^2 - 1})] = \\ &= \cosh [\operatorname{arccosh}(S)] = S \end{aligned} \tag{17}$$

Equation (8) is a direct proof for method 1, by a simple substitution. Equation (17) is an alternative proof for method 1, which explicitly shows that this choice of z makes $\sin(z)$ equal to a cosine, which in turn provides the desired real value $S \geq 1$. The values of z (7) and (11)



$$z = i \log [-i (S + \sqrt{S^2 - 1})] \quad (18)$$

$$z = \frac{\pi}{2} + i \log (S + \sqrt{S^2 - 1})$$

used in these proofs are, despite appearance, the same, as shown in (11).

Method 2

This procedure is similar to the second proof shown for Method 1. Let the desired value $S \geq 1$ be $S = \cosh(C)$. It follows that

$$C = \operatorname{arccosh}(S) = \log (S + \sqrt{S^2 - 1}) \quad (19)$$

by the definition of Inverse hyperbolic cosine. Note that C is real and non-negative. This value will be used to determine the angle z . Choose:

$$z = \frac{\pi}{2} - iC = \frac{\pi}{2} - i \log (S + \sqrt{S^2 - 1}) \quad (20)$$

Then,

$$\begin{aligned} \sin(z) &= \sin\left(\frac{\pi}{2} - iC\right) = \\ &= \frac{e^{i(\frac{\pi}{2}-iC)} - e^{-i(\frac{\pi}{2}-iC)}}{2i} = \\ &= \frac{e^{i\frac{\pi}{2}}e^{-iC} - e^{-i\frac{\pi}{2}}e^{iC}}{2i} = \\ &= \frac{ie^{-iC} + ie^{iC}}{2i} = \cosh(C) = \\ &= \cosh[\operatorname{arccosh}(S)] = S \end{aligned} \quad (21)$$

In this case, $\sin(z)$ provided not immediately a number, but the cosh of a quantity C . Relation (19) has then been used in order to directly express it as the desired number S .

Note that the value of the angle (20) used here is not the same as the one used in the second proof of Method 1, which was of the form

$$z = \frac{\pi}{2} + i \log \left(S + \sqrt{S^2 - 1} \right) \quad (22)$$

while (20) is of the form

$$z = \frac{\pi}{2} - i \log \left(S + \sqrt{S^2 - 1} \right) \quad (23)$$

First, note that the imaginary part is the same. Remembering the properties of the sine function and completing relation (12):

$$\sin \left(\frac{\pi}{2} + \alpha \right) = \sin \left(\frac{\pi}{2} - \alpha \right) = \cos(\alpha) \quad (24)$$

for any $\alpha \in \mathbb{C}$, and in particular when α is the pure imaginary quantity here presented. Method 2 is thus equivalent to Method 1. They both lead to $\cos(\alpha) = \cosh(C)$, and S is obtained from C through the arccosh in the second proof of Method 1 and in this Method 2.

Pure imaginary values

With an appropriate choice ((22) or (23)) for the complex angle z , its sine has become equal to the cosine of a pure imaginary angle:

$$\begin{aligned} \sin(z) &= \sin \left(\frac{\pi}{2} \pm iC \right) = \cos(iC) = \\ &= \cosh(C) \geq 1, \quad C \in \mathbb{R} \end{aligned} \quad (25)$$

Unlike the trigonometric sine of a real angle, it will thus assume **real** values not smaller than 1.

The drawback is that the cosine of the same complex angle will become equal to the sine of a pure imaginary angle, thus assuming only **pure imaginary** values.

Using the form (23) for z :

$$z = \frac{\pi}{2} - iC \quad (26)$$



The value of the $\cos(z)$ can be obtained from the sine itself, with an argument $\pi/2 - z$ instead of z :

$$\cos(z) = \sin\left(\frac{\pi}{2} - z\right) = \sin(iC) \quad (27)$$

Or, alternatively,

$$\cos(z) = \sin\left(\frac{\pi}{2} + z\right) = \sin(\pi - iC) = \sin(iC) \quad (28)$$

being $\sin(\pi - \alpha) = \sin \alpha$, $\forall \alpha \in \mathbb{C}$. So, either way,

$$\cos(z) = \sin(iC) = i \sinh(C) \quad (29)$$

according to the result already obtained in (6). With the choice (26) for z , while the sine is forced to be a real quantity not less than 1, the cosine is forced to be a pure imaginary quantity: **their roles**, compared to the result in (6), **have been exchanged**. Due to (19), C is a real, non-negative value and consequently the imaginary part of $\cos(z)$ is non-negative, too.

These considerations hold also when the form (22) is chosen for the complex angle z :

$$z = \frac{\pi}{2} + iC \quad (30)$$

$$\cos(z) = \sin\left(\frac{\pi}{2} + z\right) = \sin(\pi + iC) = -\sin(iC) = -i \sinh(C) \quad (31)$$

$$\cos(z) = \sin\left(\frac{\pi}{2} - z\right) = \sin(-iC) = -\sin(iC) = -i \sinh(C) \quad (32)$$

While both (22) and (23) lead to the same $\sin(z) = \cosh(C)$, the corresponding z values represent *different* angles. When different angles generate the same sine values, their cosines will necessary have opposite signs⁴, due to their position in the unit circle and the definitions themselves of sine and cosine.

Regardless of this, also the second example, just presented, showed that $\cos(z)$ is a pure imaginary number. The only difference is the sign of the imaginary part: here, it is non-positive.

⁴And vice-versa, when two different angles are related to the same cosine values.



The Curling Team

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